The Binary Right-Shift Game: Grundy Values from Young Diagrams

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Abstract: We determine the Sprague–Grundy value of every finite binary right-shift position. A move clears one chosen set bit and shifts all bits right. In a Young-diagram representation, the minimum exposed cell weight and the parity of one row width determine the four possible values $0, 1, 2, 3$. The proof gives linear-time value evaluation and winning moves for normal-play sums, and also establishes the pet property and the misère outcome criterion for sums. We also give a complete normal-play outcome criterion and a linear-time winning strategy for the variant with exactly one digit 2. Finally, we prove that each binary Grundy value has limiting probability $1/4$ for uniformly random binary strings.

Keywords: impartial games, Sprague–Grundy theory, Young diagrams, linear-time algorithms, misère play

1. Introduction

In the right-shift game, invented by the second author, a player changes one 1 to 0 in a binary string, then shifts the entire string right and discards the last digit. Leading zeros have no effect. Under normal play the player with no move loses. For example,

$$\underline{1}101 \to 010, \qquad 1\underline{1}01 \to 100, \qquad 110\underline{1} \to 110.$$

For a nonnegative integer n, choosing a set bit in zero-based position i gives

$$n \longrightarrow \left\lfloor \frac{n - 2^i}{2} \right\rfloor. \tag{1}$$

Every move decreases n. Its Grundy value is therefore defined by

$$\mathcal{G}(0) = 0, \qquad \mathcal{G}(n) = \operatorname{mex}\{\mathcal{G}(n') : n \to n'\}, \tag{2}$$

where mex is the least excluded nonnegative integer. A position is losing precisely when $\mathcal{G} = 0$. In a disjunctive sum each move changes one component; the Grundy value is the bitwise xor of the component values. The common shift prevents substrings from evolving independently.

Despite an unbounded number of options and the global shift, only four Grundy values occur. We determine them from two geometric statistics, giving linear-time evaluation and winning moves. One repeated row changes the outcome criterion but still permits a linear-time strategy. We also prove the limiting Grundy distribution under uniform input.

Related work. Young-diagram rules occur in Sato's Maya game, also known as the Sato–Welter game [5], [6], where a hook is removed and the gap closed. Abuku and Tada [7] require a second hook removal when its label multiset matches the first. Here a move deletes a whole row and shortens every other row. A single row of width three has one option here, versus three suffix options under hook removal. LCTR [3] deletes the top row or leftmost column; CRIM [4] deletes any row or column. Our move combines an arbitrary row deletion with a leftmost-column deletion. Thus the two deletions are coupled, whereas they are alternative moves in those games. MARK and MARK-t [1], [2] likewise offer subtraction and division separately.

2. Young diagrams and Grundy values

2.1 Rows, moves, and the exposed minimum

Let N be the number of ones. A one in bit position i becomes a row of width $i + 1$. Write

$$w_0 > w_1 > \cdots > w_{N-1} > 0, \qquad w_y = 0 \quad (y \geq N),$$

with the longest row at height zero. Cells are $(x, y) \in \mathbb{Z}^2_{\geq 0}$ with $x < w_y$. A move deletes one chosen row, shortens every other row by one, and discards zero widths. Thus the diagram records exactly the original moves, not just the number of ones. If row c is selected, its child widths are

$$w'_y = \begin{cases} w_y - 1, & y < c, \\ \max\{w_{y+1} - 1, 0\}, & y \geq c. \end{cases} \tag{3}$$

For a surviving row, $2y + w_y$ drops by one or three, according as its height stays fixed or decreases. This motivates weights $2y + x$, evaluated on uncovered cells in $x \leq 2y$. Call the smallest such weight the *exposed minimum*:

$$M = \min_{\substack{y \geq 0 \\ w_y \leq 2y}} (2y + w_y). \tag{4}$$

Call row y admissible when $w_y \leq 2y$. The empty row

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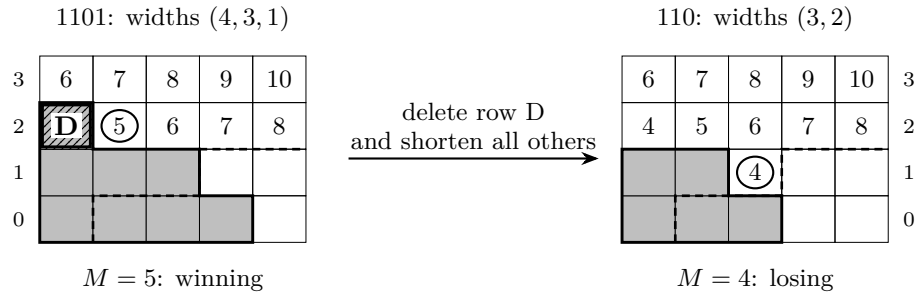


Fig. 1 From a binary string to a winning move: $1101 \to 110$. Light fill marks occupied cells; hatched row D is deleted. Numbers label uncovered cells in $x \le 2y$, with minima circled. Row height increases upward; the dashed staircase marks the weighted region.

N shows that $M \le 2N$; rows beyond N cannot improve the minimum. The empty position has $M = 0$, and every nonempty position has $M > 0$. The restriction $x \le 2y$ applies to the weights, not to the board or its legal moves. Figure 1 shows how this minimum reads a winning move.

When $M = 2T + 1$ is odd, $T < N$ since $M \le 2N$. We call row T the *critical row*.

Theorem 1 (Grundy classification)**.** *For every binary position,*

$$\mathcal{G} = \begin{cases} 0, & M \equiv 0 \pmod 4, \\ 1, & M \equiv 2 \pmod 4, \\ 2, & M \text{ odd and } w_{(M-1)/2} \text{ odd}, \\ 3, & M \text{ odd and } w_{(M-1)/2} \text{ even}. \end{cases} \tag{5}$$

In particular, a normal-play position is losing exactly when $M \equiv 0 \pmod 4$.

The proof has three steps: M drops by one to three; ordered child minima skip no even value; odd-to-odd moves reverse the critical-width parity. These facts determine the mex in (5).

2.2 How the minimum changes

Let $H_x = \#\{y : w_y > x\}$ be a column height and put

$$W(x) = 2H_x + x, \qquad R(x) = \max\{W(x), 2x\}.$$

Lemma 2. *For distinct row widths,*

$$M = \min_{x \ge 0} R(x). \tag{6}$$

Deleting row c gives

$$W'(x) = W(x+1) - 1 - 2\mathbf{1}_{\{x+1<w_c\}}. \tag{7}$$

Proof. The first uncovered weighted cell in column x has weight $Q(x) = \max\{W(x), x + 2\lceil x/2 \rceil\} \ge R(x)$. A difference is possible only for odd $x \ge 1$ with $H_x \le (x-1)/2$. Distinct widths give $H_{x-1} \le H_x + 1$, hence $Q(x-1) = R(x-1) \le 2x = R(x)$. Replacing such a column by its predecessor proves equality of the minima. A surviving row covers child column x exactly when its original width exceeds $x+1$. Thus $H'_x = H_{x+1} - \mathbf{1}_{\{x+1<w_c\}}$, proving the move formula. □

Lemma 3. *Every move from a nonempty diagram satisfies*

$$1 \le M - M' \le 3.$$

Proof. Equation (7) implies, for every $x \ge 0$,

$$R'(x) \ge R(x+1) - 3 \ge M - 3.$$

If some minimizer x_* of R is positive (even if zero also minimizes), then

$$R'(x_* - 1) \le \max\{W(x_*) - 1, 2x_* - 2\} \le M - 1.$$

If no positive minimizer exists, the minimum is at $x = 0$ and $M = 2N$. Every child has at most $N - 1$ rows, so $M' \le 2N - 2 < M$. Together these bounds prove the claim. □

Lemma 4. *Among the children, deleting the largest width minimizes the exposed minimum and deleting the smallest maximizes it. Denote these values by $M_{\min}$ and $M_{\max}$. Then*

$$M_{\min} \le M - 2 \le M_{\max}. \tag{8}$$

If M is odd, more precisely,

$$M_{\min} = M - 3, \qquad M_{\max} = M - 1. \tag{9}$$

Proof. By the row deletion rule, the child diagrams increase by inclusion as the deleted row index increases. The exposed minimum is monotone under inclusion, proving the first assertion.

If $M = 2N$, the row-count bound gives $M_{\min} \le M - 2$. Otherwise a minimizer $x_* > 0$ exists. If $x_* < w_0$, deleting w_0 gives

$$R'(x_* - 1) \le \max\{M - 3, M - 2\} = M - 2.$$

If $x_* \ge w_0$, we may choose $x_* = w_0$, since $R(x) = 2x$ in that region. Child column $x_* - 1$ is then empty, giving $R'(x_* - 1) = 2x_* - 2 = M - 2$. When M is odd, a minimizer has $W(x_*) = M$, $2x_* \le M - 1$, and $x_* < w_0$. The same argument strengthens the upper bound to $M - 3$, which is exact by Lemma 3.

For the largest child put $\ell = w_{N-1}$. If $x < \ell - 1$, then $H_{x+1} = N$ and

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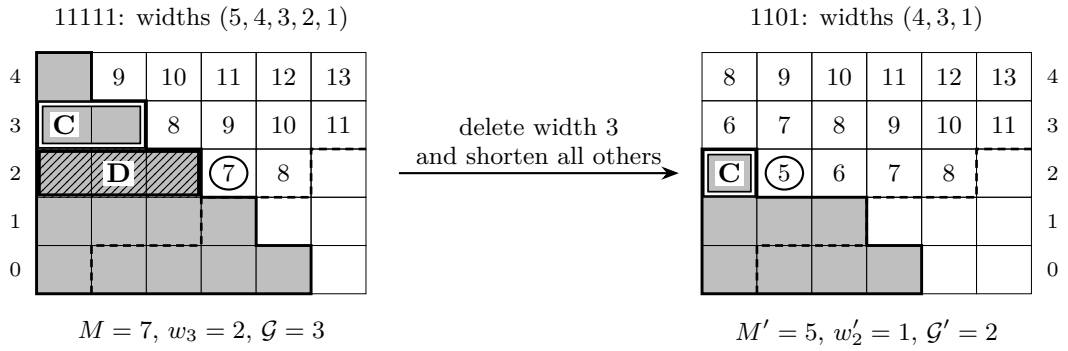


Fig. 2 The move 11111 → 1101. Labels beside the grids give row heights. Hatched row D is deleted. The double-framed critical row C moves down-left and its width decreases from 2 to 1. Light fill marks occupied cells; circles mark minima and the dashed staircase bounds $x \le 2y$.

$$W'(x) = 2N - 2 + x \ge M - 2.$$

If $x \ge \ell - 1$, equation (7) gives $R'(x) \ge R(x+1) - 2 \ge M - 2$. For odd M, the first bound is at least $M - 1$ because $M \le 2N - 1$. In the second region, either $W(x+1) \ge M$, giving $W'(x) \ge M - 1$, or $2(x+1) \ge M + 1$, giving $2x \ge M - 1$. Thus $M_{\max} \ge M - 1$, with equality by Lemma 3. □

Lemma 5. *As the deleted row changes from largest to smallest, the nondecreasing sequence of child exposed minima does not skip an even integer. Consequently, every nonempty diagram with even M has a child with exposed minimum $M - 2$.*

Proof. For adjacent choices c and $c+1$, column heights differ by at most one. Thus their R values differ pointwise by at most two, and their minima differ by at most two. Suppose that an even value k were skipped: the two minima would be $k-1$ and $k+1$. Only row c differs between the two children, with widths $w_{c+1} - 1$ and $w_c - 1$. Only in that row can a cell exposed in the smaller child be covered in the larger one. Thus its minimum $k-1$ occurs there, so

$$2c + w_{c+1} - 1 = k - 1, \qquad w_{c+1} - 1 \le 2c.$$

Thus $w_{c+1} = k - 2c$ is even and at least two: the case $w_{c+1} = 1$, in which the row disappears, is excluded. In both children the next row has width

$$\max\{w_{c+2} - 1, 0\} \le w_{c+1} - 2.$$

This row is admissible and exposes a value at most $2(c+1) + w_{c+1} - 2 = k$, contradicting the larger exposed minimum $k+1$. This argument includes $c = N - 2$, since $w_N = 0$. Finally, the even value $M - 2$ lies between the two extremal child minima by Lemma 4, so it must occur. □

2.3 Critical-row transitions

Figure 2 illustrates the parity reversal established next.

Lemma 6. *Suppose $M = 2T + 1$ is odd. Every odd-minimum child satisfies $M' = M - 2$, its selected row has index $c < T$, and*

$$w'_{T-1} = w_T - 1. \tag{10}$$

Such a child exists if and only if $w_T > 1$.

Proof. Lemma 3 gives $M' = 2T - 1$. Let row y attain this child minimum. If $y < c$, then $w_y - 1 = 2T - 1 - 2y$ is odd. Admissibility and evenness give $w_y \le 2y$, contradicting $M \le 2y + w_y = 2T$. Since $2y \le 2T - 1$, we obtain $c \le y < T$, and (3) gives (10). Its width is positive, proving necessity.

For sufficiency, assume $w_T > 1$ and put $f(y) = 2y + w_y$. Let r be the largest admissible index with $f(r) = M$. Then $r < T$: $f(T) > M$ and $f(y) > M$ for $y > T$. Row $r+1$ is admissible, and strict widths give $f(r+1) \le f(r) + 1$. By the choice of r it equals $M + 1$. Delete row r. Since $w_{r+1} \ge w_T > 1$, its new row r is positive and exposes $f(r+1) - 3 = M - 2$. Every admissible child row $y \ge r$ comes from an admissible parent row $y + 1 > r$ of weight greater than M, reduced by at most three. For $y < r$, an already admissible row has weight at least $M - 1$. A newly admissible row had $w_y = 2y + 1$ and now has weight $4y$; its next parent row is admissible, so $M \le 2(y+1) + w_{y+1} \le 4y + 2$. Thus all child weights are at least $M - 2$, proving equality. □

Proof of Theorem 1. Induct on M, starting with the empty position. Every move lowers M by Lemma 3. If M is even, every even child has minimum $M - 2$ and hence the opposite value in $\{0, 1\}$; odd children have values 2 or 3. Lemma 5 supplies an even child, so the mex is $(M/2) \bmod 2$.

If $M = 2T + 1$, Lemma 4 gives even children at $M - 1$ and $M - 3$, with values 0 and 1. By Lemma 6, every odd child has the opposite critical width parity. If w_T is odd, no child has value 2, giving mex 2. If w_T is even, an odd child exists and has value 2; no child has value 3, giving mex 3. □

For example, 11111 has $M = 7$ and critical width 2, so its value is 3. The move in Fig. 2 reduces the critical width to 1 and reaches value 2.

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3. Linear-time evaluation and strategy

Scanning a length-L binary string constructs its width array in $O(L)$ time. Equation (4), including the empty row N, computes M in $O(N)$ time, and one array lookup gives the critical width. We can also evaluate all options in $O(N)$ time. Put $F(y,u) = 2y + u$ if $u \le 2y$, and $+\infty$ otherwise, and, for $0 \le y < N$, set

$$A_y = F(y, w_y - 1), \qquad B_y = F(y, \max\{w_{y+1} - 1, 0\}).$$

After deleting row c, equation (3) gives

$$M_c = \min\left\{\min_{y<c} A_y,\ \min_{c\le y<N} B_y\right\}. \tag{11}$$

All child rows through index $N-1$ are represented, including the first empty row; later empty rows cannot improve the minimum. Prefix and suffix minima compute every M_c in linear time. If M_c is odd, put $T_c = (M_c - 1)/2$. Its critical width is

$$u_c = \begin{cases} w_{T_c} - 1, & T_c < c, \\ w_{T_c+1} - 1, & T_c \ge c. \end{cases} \tag{12}$$

It is positive by the critical-row existence argument. Thus each child value follows from (5) in constant time.

Theorem 7. *For a normal-play disjunctive sum of binary right-shift positions, given as nonempty strings of total length L, the Grundy value and a winning move, when one exists, can be found in $O(L)$ time and space.*

Proof. Compute the xor s of component values. If $s \neq 0$, choose a value g with $g \oplus s < g$ and scan that board's options for the guaranteed target $g \oplus s$. The resulting xor is zero. There are at most L components, and all scans together take $O(L)$ time and space. □

The model is a RAM with $O(\log L)$-bit words and explicit digit strings; zero is entered as 0. Widths, indices, and minima are $O(L)$, so $2L+1$ suffices as the infinity sentinel. For 1101, the children $10, 100, 110$ have values $1, 1, 0$; clearing the rightmost one is its only winning move.

4. Misère play and pet games

Under misère play the last mover loses. Set $\mathcal{G}^-(0) = 1$ and use the usual mex recursion elsewhere; then $\mathcal{G}^- = 0$ characterizes single-board misère $\mathcal{P}$-positions. A game is *pet* if every pair $(\mathcal{G}, \mathcal{G}^-)$ is $(0,1)$, $(1,0)$, or (k,k) for $k \ge 2$ [8]. Every nonterminal value-0 position has a child with minimum $M - 2$ by Lemma 5, hence a value-1 child. This gives a direct proof.

Theorem 8. *The binary right-shift game is pet, with*

$$\mathcal{G}^-(n) = \begin{cases} 1, & \mathcal{G}(n) = 0, \\ 0, & \mathcal{G}(n) = 1, \\ \mathcal{G}(n), & \mathcal{G}(n) \ge 2. \end{cases} \tag{13}$$

Its single-board misère $\mathcal{P}$-positions are exactly those with $M \equiv 2 \pmod 4$.

Proof. Induct over the finite game. The terminal case has pair $(0,1)$. At a nonterminal value-0 position no option has normal value 0, and the preceding escape property supplies an option of value 1. By induction the misère option values therefore contain 0 but not 1, giving mex 1. At a normal value-1 position no option has normal value 1; by induction no misère option has value 0, giving mex 0. At a position of normal value 2 or 3, the normal option values contain 0 and 1 and every other smaller value, while omitting the position's own value. Interchanging 0 and 1 in those option values does not change their mex. This proves (13). □

Misère sums. The pet property also solves sums of binary boards. Put $g_i = \mathcal{G}(B_i)$. The sum is a misère $\mathcal{P}$-position exactly when the number of $g_i = 1$ is odd, if all $g_i \le 1$, or when $\bigoplus_i g_i = 0$, otherwise. Indeed, the sum theorem for tame games [8] says that the normal and misère values are interchanged between 0 and 1 precisely when every component is a swap position; elsewhere the two values agree. Here the swap positions are exactly those with $g_i \le 1$. The criterion is evaluable in $O(L)$ time.

5. A single digit two

Allow digits $0, 1, 2$, with exactly one 2. A move decreases a positive digit by one, then discards the rightmost digit. Let c_i be the digit at zero-based position i from the right. It contributes c_i rows of width $i+1$; write $N = \sum_i c_i$ for the row count. Thus there is one repeated width w, at ranks $t, t+1$, where t counts the longer rows. Deleting either copy gives the same binary child A; write $\alpha = M(A)$. A *long* deletion selects a width above w, and a *short* deletion one below w. For $w > 1$ they preserve a repeated width $w-1$, at rank $t-1$ and t, respectively. For $w = 1$ all children are binary.

The exposed-cell minimum still has the row interpretation (4), now with weakly decreasing widths. For columns put

$$\begin{aligned} W_x &= 2\sum_{i\ge x} c_i + x, & b_x &= x + 2\lceil x/2 \rceil, \\ Q_x &= \max(W_x, b_x), & M &= \min_{x\ge 0} Q_x, \\ H &= \min_{0\le x<w} Q_x, & D &= \lfloor M/2 \rfloor. \end{aligned} \tag{14}$$

Indeed the lowest exposed admissible cell in column x has height $\max(\sum_{i\ge x} c_i, \lceil x/2 \rceil)$. Minima may be restricted to $0 \le x \le L$. Define the geometric flag

$$\sigma = (H > M) \ \wedge\ (t < D) \ \wedge\ (w + D \text{ odd}). \tag{15}$$

These conditions put all minima in $x \ge w$, the lower repeated row below height D, and w, D at opposite parities. They govern the location, rank and parity of losing children; A gives an escape if $\alpha \equiv 0 \pmod 4$.

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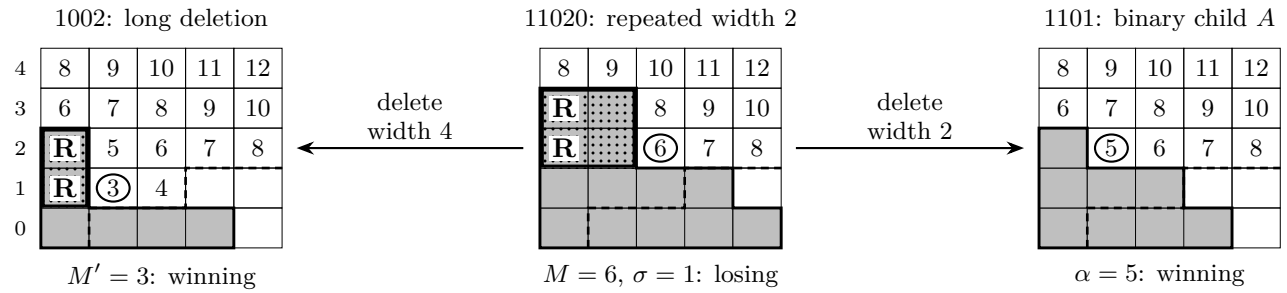


Fig. 3 Dotted rows R form the repeated pair, which survives a long deletion. The last long deletion gives minimum 3; deleting R instead gives 5, skipping 4. Here $H = 8$, $w = 2$, $t = 2$, $D = 3$, so $\sigma = 1$ and $\alpha = 5 = M - 1$: the parent loses despite $M \equiv 2 \pmod 4$. Circles mark exposed minima.

Theorem 9. *The normal-play losing positions with one digit* 2 *are exactly*

$M \bmod 4$	*condition for* $\mathcal{P}$
0	$\neg\sigma$
1	*never*
2	σ *and* $\alpha = M - 1$
3	σ

The outcome and a winning move, when one exists, are computable in $O(L)$ time and space on a single board.

This is an outcome theorem for one board. Full one-2 Grundy values, sums and misère play are outside its scope. Already 111120010 has option values $1, 1, 1, 3, 2, 0$, so its Grundy value is 4.

The no-even-skip property in Lemma 5 can now fail (Fig. 3). We locate the exceptional jump, then track each child's flag by splitting at the repeated width. Keep the row notation w_y, now weakly decreasing and zero for $y \geq N$. A deletion of width z satisfies

$$W'_j = W_{j+1} - 1 - 2\mathbf{1}_{\{j<z-1\}}. \tag{16}$$

Since $b_{j+1} - b_j \in \{1, 3\}$, every child minimum satisfies

$$M - 3 \leq M' \leq M - 1. \tag{17}$$

The lower bound follows column by column. A positive minimizing column shifts to a value at most $M-1$; if only column zero minimizes, then $M = 2N$ and the child has at most $N - 1$ rows, proving the upper bound.

Lemma 10. *Child minima increase as the selected row moves from longest to shortest, with adjacent differences at most two. An even value k can be skipped only between the last long deletion and A, and then $2t + w = k + 2$. Furthermore:*
(1) if $M \equiv 2 \pmod 4$, the child minima bracket $M - 2$;
(2) if $M \equiv 1 \pmod 4$, the shortest deletion gives $M - 1$;
(3) if M is odd, the longest deletion gives $M - 3$.

Proof. Inclusion orders the children; adjacent column walks differ by at most two. If adjacent deletions $c, c+1$ skip even k, their minima are $k-1, k+1$. Only row c differs, so its smaller width satisfies $2c + w_{c+1} - 1 = k - 1$, $w_{c+1} - 1 \leq 2c$. Thus $w_{c+1} = k - 2c$ is positive and even. Unless $w_{c+2} = w_{c+1}$, the common next row has width at most $w_{c+1} - 2$ and exposed weight at most k, a contradiction. Equality requires $c = t - 1$, giving the asserted exception.

For (1), a minimizing column zero gives a child at most $M - 2$ by row count. Otherwise a minimizing x is positive and even, with $b_x \leq M - 2$. It lies below the largest width, since an empty even column has value divisible by four. Deleting the largest width gives at $x - 1$ a walk smaller by three and boundary smaller by one, hence a value at most $M - 3$. Delete the smallest width z for the other bound. At $j < z - 1$ its walk is $2N - 2 + j \geq M - 2$. Else put $x = j + 1$. Either $W_x \geq M$, giving $W'_j \geq M - 1$, or $b_x \geq M$. An even x loses one from its boundary; an odd x has $b_x \geq M + 1$ before losing three. Both give $M - 2$.

For (2), the same shortest deletion has affected walk at least $2N - 2 \geq M - 1$. In the unaffected region $W_x \geq M$ suffices; otherwise an even x has $b_x \geq M + 3$ before losing one, and an odd x has $b_x \geq M + 2$ before losing three. Use (17).

For (3), a minimizing x is odd. Below the largest width its walk and boundary both lose three. Otherwise it must equal the largest width, and $M = 2x + 1$. At $x - 1$, whose boundary is $2x - 2$, minimality forces $2q \geq x + 2$, where $q \leq 2$ is the largest-width multiplicity. Only $x = 1, q = 2$ is possible; that board has $M = 3$ and an empty child. □

Assume $w \geq 2$ and split A at column $w - 1$:

$$U = \min_{1 \leq x < w} \max(W_x - 3, b_{x-1}), \tag{18}$$
$$V = \min_{x \geq w} \max(W_x - 1, b_{x-1}).$$

Then $\alpha = \min(U, V)$. A long child has $H' = U$; a short child has left minimum $H' = L_c$ and fixed right minimum V. These identities determine the location test $H' > M'$; the deletion type gives $t' = t - 1, t$, and M' gives $D' = \lfloor M'/2 \rfloor$ for the rank and parity tests. Starting at A, the L_c increase and cannot skip an even value: the preceding skip proof works restricted to columns below $w-1$, since both its differing smaller row and common next-row witness have shifted width below $w-1$. No equal pair occurs among the short rows. The first value is U.

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Lemma 11. *If σ holds, then $\alpha \geq M-2$. For $M \equiv 0 \pmod 4$ one always has $V \geq M-1$, with equality when $H > M$.*

Proof. Left columns have $Q_x \geq M+1$, so $U \geq M-2$. If $V \leq M-3$, some $x \geq w$ must have $W_x \leq M-2$ and $2x-2 \leq M-3$. Since $Q_x \geq M$, these force odd M, odd $x = (M-1)/2$, and $b_x = M$. If $x > w$, width x is not duplicated, so $W_{x-1} \leq W_x + 1 \leq M-1$ and $b_{x-1} = M-3$, a contradiction. If $x = w$, the flag's parity fails.

For the second assertion, a right value at most $M-2$ would require $b_{x-1} \leq M-2$ and $W_x \leq M-1$. Since $b_{D-1} = M-1$, this forces $x \leq D-1$, hence $Q_x < M$, a contradiction. If $H > M$, a right minimizing parent column shifts to at most $M-1$. □

Proof of Theorem 9. Induct on M, using (17); binary children use Theorem 1. We show that predicted losing positions have only winning children, and construct a losing child in every other case. First let $w = 1$. All children are binary. At residue zero none can have minimum divisible by four. At residues one, two, three Lemma 10 gives a child of minimum $M-1, M-2, M-3$, respectively; the even skip is impossible for odd w. The flag is false at residue zero: $t = N-2$ and $M \leq Q_1 = \max(2t+1, 3)$ imply $t \geq D$. At residues two and three its parity fails. This proves the boundary case. Below $w \geq 2$.

By Lemma 11, a flagged residue-three board has $\alpha \in \{M-2, M-1\}$, so A is winning. On a flagged residue-two board, A is winning exactly when $\alpha = M-1$. At residue zero, A is always winning by (17). In each residue case, (17) allows at most three child minima. The split above then lets us test the flag for each candidate losing child.

Residue one: always winning. Put $k = M-1$. The largest child minimum is k. If $\alpha = k$, play A. Otherwise $\alpha < k$, so a short child reaches k, forcing $V \geq k$ and $U = \alpha < k$. The short-left sequence hits even k. Choose $L_c = k$; its minimum is attained on the left, so its flag is false and it is losing.

Residue two: losing iff σ and $\alpha = M-1$. Put $k = M-2$; D is odd. A losing child can only have minimum k with false flag, or $M-3$ with true flag. For odd w, the even skip is impossible, so a k-child exists. A binary one is losing; a preserving one has both $w' = w-1$ and $D' = D-1$ even, making it unflagged and losing.

Let w be even. A preserving child of minimum $M-3$ has both $w' = w-1$ and $D' = D-2$ odd, so is unflagged. Thus only unflagged k-children matter. If $\alpha = k$, play A. If σ holds and $\alpha \neq k$, Lemma 11 and (17) force $U, V \geq \alpha = M-1$. No short child reaches k; every long k-child has left minimum $U > k$, rank $t-1 < D-1$, and the required parity. It is flagged and winning, so the parent is losing.

If σ fails and $\alpha \neq k$, then $\alpha = M-1$ or $M-3$. In the first case $H > M$: a left minimizing column zero gives $\alpha \leq M-2$, while a positive left minimizing column is even with $b_x \leq M-2$, giving $U \leq M-3$. Both contradict $\alpha = M-1$. The failed flag therefore means $t \geq D$; the skip condition $2t + w = M$ is impossible. A k-child exists before A, hence is long. Its rank $t-1 \geq D-1$ makes it unflagged and losing. In the second case a short child reaches at least k by Lemma 10(1), giving $V \geq k$ and $U = k-1$. Choose $L_c = k$; again its left minimum makes it losing.

Residue three: losing iff σ. Put $k = M-3$; D is odd. A k-child exists. For odd w it is losing: either it is binary, or its repeated width and D' are even, excluding the flag. For even w, preserving children of minimum $M-1$ have odd width and odd D', and children of minimum $M-2$ have residue one; none is losing. If σ holds, $U, V \geq M-2 > k$. Only long children can reach k; their left minimum is larger, their rank $t-1 < D-1$, and their parity holds. They are flagged and winning. Thus the parent is losing. If σ fails, play A when $\alpha = k$. Otherwise every k-child is long. Necessarily $t \geq D$: if $t < D$, failure of the flag forces $H = M$, and an odd left minimizing column gives $U \leq M-3 = k$, contrary to $\alpha > k$. Hence a k-child fails its rank condition and is losing.

Residue zero: losing iff $\neg\sigma$. Here D is even, and A is winning. A losing preserving child must be flagged, with minimum $M-2$ or $M-1$ and $D' = D-1$. Its parity requires odd w; even w therefore gives a losing parent. Suppose w is odd.

If σ fails, either $t \geq D$, excluding every child's rank condition, or $H = M$. In the latter case choose an even left minimizing column $x \leq D$. If $x = D$, then $w > D$ and all child minimizing columns lie at most $D-1 < w-1$. If $x = 0$, each child has left value at most $2N-2 = M-2$. Otherwise $0 < x \leq D-2$: long children have left value at most $M-3$ at $x-1$, and short children have left value at most $M-1$, with right minimum $V \geq M-1$. In each case a child with either relevant minimum has a left minimum, so is unflagged. The parent loses.

Finally let σ hold. Then $N \geq D+1$, $t < D$, $3 \leq w \leq D-1$, and $V = M-1$. If $t \leq D-2$, a short row exists; delete the smallest width z. Its left minimum is at least M: for $1 \leq x < w$, if $x < z$ the shifted walk is $2N+x-3 \geq M$; if $x \geq z$, a shifted value at most $M-1$ would force $x \leq D$ and $W_x \leq M$, contradicting $H > M$. The right minimum is $M-1$; the child's rank $t < D-1$ and parity hold. It is flagged at residue three, hence losing.

The remaining case is $t = D-1$. Delete the shortest long row, of width $z > w$. For $1 \leq x < w$ its walk is at least M; for $w \leq x < z$ it is $M + x - 5 \geq M-2$; at $x \geq z$ the right expression is at least $M-1$. The child has minimum $n \in \{M-2, M-1\}$, attained on the right, and rank $D-2 < D-1$, so its flag holds. Residue three ($n = M-1$) is losing. For $n = M-2$, the residue-two criterion also requires $\alpha' = n-1$. This extra condition concerns the child's binary option, so we inspect that grandchild. At column $x-2$, $x \geq 2$, its walk and binary boundary are

$$W_x - 2 - 2\mathbf{1}_{\{x<z\}} - 2\mathbf{1}_{\{x<w\}}, \qquad 2x-4.$$

For $x < w$ the walk is at least $M + x - 4 \geq M-2$; for

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$w \le x < z$ it is $M + x - 6 \ge M - 3$. For $x \ge z$, a value at most $M - 4$ would force $x \le D$, $W_x \le M - 2$, hence $x = D$ by $Q_x \ge M$. Width D is not duplicated, so $W_{D-1} \le W_D + 1 \le M - 1$ and $b_{D-1} = M - 1$, a contradiction. The binary grandchild's minimum is therefore at least $M - 3$, and by decrease exactly $M - 3 = n - 1$. The residue-two child is losing as well. This completes the induction.

For the algorithm, compute M, H, w, t by a column scan, using $W_{x+1} = W_x + 1 - 2c_x$, and compute α on A. For a winning board, first play A if $\alpha \equiv 0 \pmod 4$. Otherwise put $k = 4\lfloor M/4 \rfloor$ and use the following choices; the parent is assumed winning. Rows after the first apply to $w > 1$.

Case	Selected deletion
$w = 1$	child minimum k
$M \equiv 1$	short, $L_c = k$
$M \equiv 2$, w odd	child minimum k
$M \equiv 2$, w even, $\alpha = M - 1$	long, child minimum k
$M \equiv 2$, w even, $\alpha = M - 3$	short, $L_c = k$
$M \equiv 3$	child minimum k
$M \equiv 0$, $t \le D - 2$	smallest width
$M \equiv 0$, $t = D - 1$	shortest long row

Residues in the table are modulo four. Each choice was proved above to exist and to reach a losing position. The prefix/suffix scan (11) also works for repeated widths. To compute every L_c, restrict its row candidates further to width below $w - 1$. Thus the searches take $O(L)$ time and space. Repeating this procedure after each opposing move, switching to the binary strategy when the 2 disappears, gives a winning strategy. □

6. Limiting distribution

Theorem 12. *Choose a binary string uniformly from the 2^L strings of length L, allowing leading zeros. Then, for each $j \in \{0, 1, 2, 3\}$,*

$$\lim_{L\to\infty} \mathbb{P}(\mathcal{G} = j) = \frac{1}{4}.$$

The proof first localizes the exposed minimum to an upper-bit block, whose random walk makes its residues modulo four asymptotically uniform. An independent lower-bit block then makes the critical-width parity asymptotically fair. We use a simple symmetric walk: independent increments 1 and -1, each with probability $1/2$, starting at zero.

Lemma 13. *Let S be a simple symmetric walk and $\mathcal{R}_n = \max_{0\le i\le n} S_i$. There is an absolute constant C such that*

$$\mathbb{P}(\mathcal{R}_n \le u) \le \frac{C(u+1)}{\sqrt{n+1}} \quad (u \ge 0), \tag{19}$$

$$\mathbb{E}\mathcal{R}_n \le \sqrt{n}, \tag{20}$$

$$\mathbb{P}\left(\max_{i\le n} |S_i| \ge v\right) \le \frac{n}{v^2} \quad (v > 0). \tag{21}$$

Moreover,

$$\mathbb{E}(-1)^{\mathcal{R}_n} = \mathbb{P}(S_n = 0). \tag{22}$$

If $\mathcal{R}_n$ and $\mathcal{R}_d$ are independent, then

$$\mathbb{P}(\mathcal{R}_n \le \mathcal{R}_d) \le \frac{C(\sqrt{d}+1)}{\sqrt{n+1}}. \tag{23}$$

Proof. Reflecting a path after its first visit to $k \ge 1$ gives

$$\mathbb{P}(\mathcal{R}_n \ge k) = \mathbb{P}(S_n \ge k) + \mathbb{P}(S_n \ge k+1).$$

Subtract consecutive tails; the case $k = 0$ also follows by symmetry. Thus, for every integer $k \ge 0$,

$$\mathbb{P}(\mathcal{R}_n = k) = \mathbb{P}(S_n = k) + \mathbb{P}(S_n = k+1). \tag{24}$$

The central-binomial estimate $\max_k \mathbb{P}(S_n = k) \le C/\sqrt{n+1}$ proves (19). Summing the reflected tails and using symmetry gives $\mathbb{E}\mathcal{R}_n \le \mathbb{E}|S_n| \le \sqrt{\mathbb{E}S_n^2} = \sqrt{n}$.

For (21), stop at the first time τ with $|S_\tau| \ge v$. On $\{\tau \le n\}$, independent future increments give $\mathbb{E}(S_n^2 \mid \mathcal{F}_\tau) = S_\tau^2 + n - \tau \ge v^2$. Hence $n = \mathbb{E}S_n^2 \ge v^2\mathbb{P}(\tau \le n)$. Alternating (24) over $k \ge 0$ telescopes to $\mathbb{P}(S_n = 0)$, which vanishes for odd n; this proves (22). Finally condition on $\mathcal{R}_d$ in (19) and use (20) to obtain (23). □

Proof of Theorem 12. Read the bits from most significant to least significant and let Z_n be the number of ones minus the number of zeros in the first n bits. This is a simple symmetric walk. In column x, the height H_x is the number of ones among the first $L - x$ bits. Substituting $n = L - x$ into (6) yields the exact identity

$$M = L + K, \qquad K = \min_{0\le n\le L} \max\{Z_n, L - 2n\}. \tag{25}$$

Localizing the minimum. Set

$$a = \lfloor L^{2/3} \rfloor, \quad c = \lfloor L/2 \rfloor - a, \quad b = \lfloor L/2 \rfloor + a,$$

and $e = L - a$, $N = e - b$. For sufficiently large L these satisfy $0 < c < b < e < L$ and $N \sim L/2$. Define

$$K_* = \min_{b\le n\le e} Z_n, \qquad M_* = L + K_*.$$

We show that $K = K_*$ with probability tending to one.

By (21), the event $E_0 = \{\max_{n\le L} |Z_n| \le a\}$ fails with probability at most L/a^2. On E_0, the line $L - 2n$ is above K_* for $n < c$, and below Z_n for $n \ge b$. To handle $[c, b]$, put

$$U = Z_b - \min_{c\le n\le b} Z_n, \qquad V = Z_b - K_*.$$

Time reversal shows that U has the law of $\mathcal{R}_{b-c}$, while V has the law of $\mathcal{R}_N$. They depend on disjoint increments and are independent. If $V > U$, the central minimum is strictly below every walk value in $[c, b]$. By (23), the probability that this fails is $O(\sqrt{a/L})$. Similarly, let

$$D = Z_e - K_*, \qquad F = Z_e - \min_{e\le n\le L} Z_n.$$

Here D and F are independent with laws $\mathcal{R}_N$ and $\mathcal{R}_a$. The event $D > F$ puts the central minimum below all values in $[e, L]$, and again has failure probability $O(\sqrt{a/L})$. On the

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intersection of these three events, the minimum in (25) is exactly K_*. Consequently,

$$\mathbb{P}(K \neq K_*) = O\left(\frac{L}{a^2} + \sqrt{\frac{a}{L}}\right) = o(1). \tag{26}$$

The surrogate M_* depends only on the first e bits, and is independent of the remaining a bits.

Residues modulo four. Write $K_* = Z_b + J_N$, where $J_N = \min_{0 \le i \le N}(Z_{b+i} - Z_b)$ is independent of Z_b and has the law of $-\mathcal{R}_N$. For $b \ge 1$, the residue of Z_b modulo four is exactly uniform on the two residues of parity b: condition on all but one increment, and the two possibilities for that increment differ by two modulo four. Thus for every residue r,

$$\begin{aligned}\mathbb{P}(M_* \equiv r \pmod 4) &= \tfrac{1}{2}\mathbb{P}(L + b + J_N \equiv r \pmod 2)\\ &= \tfrac{1}{4} + O(N^{-1/2}),\end{aligned}$$

where the last equality follows from (22) and the binomial bound. By (26), M has the same limiting residue distribution. Theorem 1 now gives the required limits for $\mathcal{G} = 0$ and $\mathcal{G} = 1$, and $\mathbb{P}(M_* \text{ odd}) \to 1/2$.

Parity of the critical width. Condition on the first e bits. They determine M_*, D, and their number of ones $A_e = (e + Z_e)/2$. If $M_* = 2T_* + 1$ is odd, put

$$r = T_* - A_e + 1 = \frac{a - D + 1}{2}. \tag{27}$$

When $r > 0$, the row with index T_* is the r-th one in the remaining low block, provided that the block contains that many ones. Let $q = \lfloor L^{5/12} \rfloor$. Since D has the law of $\mathcal{R}_N$, (19) and (21) give

$$\begin{aligned}\mathbb{P}(D < q) &= O(q/\sqrt{N}) = o(1),\\ \mathbb{P}(D > a/2) &= O(N/a^2) = o(1).\end{aligned}$$

Call upper-bit realizations with $q \le D \le a/2$ good. On a good realization with odd M_*,

$$\frac{a}{4} \le r \le \frac{a - q + 1}{2}.$$

Extend the independent low-bit sequence by further independent fair bits and let τ_r be the position of its r-th one. It is the sum of r independent geometric waiting times Y with $\mathbb{P}(Y = k) = 2^{-k}$, $k \ge 1$. Hence

$$\mathbb{E}(-1)^Y = -\frac{1}{3}, \qquad |\mathbb{E}(-1)^{\tau_r}| = 3^{-r}.$$

The two parities of τ_r therefore have probabilities $1/2 \pm 3^{-r}/2$. Moreover, $\tau_r > a$ is equivalent to a binomial random variable with parameters $(a, 1/2)$ being smaller than r. Its variance is $a/4$, so Chebyshev's inequality and (27) imply, on every good realization,

$$\mathbb{P}(\tau_r > a \mid \text{upper bits}) \le \frac{a}{(D-1)^2} \le \frac{a}{(q-1)^2} = o(1).$$

Whenever $\tau_r \le a$, row T_* has width $a + 1 - \tau_r$. For $\epsilon \in \{0, 1\}$ put

$$E_\epsilon = \{\tau_r \le a,\ a + 1 - \tau_r \equiv \epsilon \pmod 2\}.$$

For each good upper-bit realization with odd M_*, removing the truncation event from the geometric parity event gives

$$\begin{aligned}&\left|\mathbb{P}(E_\epsilon \mid \text{upper bits}) - \tfrac{1}{2}\right|\\ &\qquad \le \frac{3^{-r}}{2} + \mathbb{P}(\tau_r > a \mid \text{upper bits}) = o(1).\end{aligned}$$

The error is uniform, since $3^{-r} \le 3^{-a/4}$ and $a/(q-1)^2 \to 0$.

Finally discard the events of a bad upper block, $M \neq M_*$, and $\tau_r > a$ on good odd upper blocks. All have probability tending to zero. For the remaining odd cases, $T = T_*$ and the width just found is the one in Theorem 1. For $j = 2, 3$, summing the conditional probabilities gives

$$\mathbb{P}(\mathcal{G} = j) = \tfrac{1}{2}\mathbb{P}(M_* \text{ odd}) + o(1) \longrightarrow \tfrac{1}{4}.$$

This proves all four limits. □

7. Verification

Code, reproduction commands and Lean proofs are available in the project repository.[*1] Literal-move mex checks agree with the binary formula, every option value, and the critical-row construction on all 2^{20} length-20 inputs and 10,485,760 moves. Independent outcome DP checks all $20 \cdot 2^{19}$ length-20 one-2 strings, including winning moves for all 7,156,324 winning inputs; literal ternary DP cross-checks every one-2 string through length 12.

Lean 4.19.0 developments certify the binary formula, pet identity and distribution limit using earlier proofs. The current proof organization, one-2 theorem and misère-sum application have not been formalized. The repository documents the certificate map and the cost-model qualifications for the strategy implementation.

References

[1] A. S. Fraenkel: Aperiodic Subtraction Games, *The Electronic Journal of Combinatorics*, Vol. 18, No. 2, #P19 (2011).

[2] A. Guo: Winning Strategies for Aperiodic Subtraction Games, *Theoretical Computer Science*, Vol. 421, pp. 70–73 (2012).

[3] E. Gottlieb, M. Krnc, and P. Muršič: Sprague–Grundy Values and Complexity for LCTR, *Discrete Applied Mathematics*, Vol. 346, pp. 154–169 (2024).

[4] I. Bašić, E. Gottlieb, and M. Krnc: CRIM: A Natural Game on Integer Partitions, arXiv:2606.16828v1 (2026).

[5] M. Sato (notes by H. Enomoto): Mathematical Theory of Maya Game (in Japanese), *RIMS Kôkyûroku*, Vol. 98, pp. 105–135 (1970).

[6] N. Kawanaka: Sato–Welter Game and Kac–Moody Lie Algebras, *RIMS Kôkyûroku*, Vol. 1190, pp. 95–106 (2001).

[7] T. Abuku and M. Tada: A Multiple Hook Removing Game Whose Starting Position is a Rectangular Young Diagram with Unimodal Numbering, *Integers*, Vol. 23, #G1 (2023).

[8] V. Gurvich and N. B. Ho: On Tame, Pet, Domestic, and Miserable Impartial Games, *Discrete Applied Mathematics*, Vol. 243, pp. 54–72 (2018).

[*1] `https://github.com/ogiekako/right_shift_game`

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