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Analysis of an Edge Coloring Algorithm Using Chernoff Bounds

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1 Introduction

Given a multigraph $G = (V, E)$ with n vertices and m edges and a color set $\mathcal{C} = \{1, 2, \ldots, k\}$, the nearly equitable edge coloring is an assignment of given colors to edges in G such that, among the edges incident to each vertex, the numbers of edges colored with any two colors differ by at most two. The notion of the nearly equitable edge coloring was introduced in 1982 by Hilton and de Werra [1], who also proved that any graph has a nearly equitable edge coloring. Their proof is constructive and easily leads to an algorithm for finding such a coloring in $O(km^2)$ time as mentioned in [2]. Later, Nakano et al. [2] showed an algorithm that runs in $O(m^2/k+mn)$ time. In 2004, Xie et al. [3] presented a more efficient algorithm, which improved the running time to $O(m^2/k)$ and moreover satisfied the following *balanced constraint*: The numbers of the edges colored with any two colors differ by at most one.

The previous algorithm presented by Xie et al. works as follows: Initially assign k colors $1, 2, \ldots, k$ to k uncolored edges repeatedly until all the edges are colored, and then invokes an algorithm called Recolor to modify the current edge coloring whenever it is not nearly equitable. Hence, the running time of the algorithm is decided by the running time of Recolor and the number of calls to Recolor. To analyze the number of calls to Recolor, they introduced a potential Φ_π, and showed that Recolor always runs in $O(|E_\pi(i) \cup E_\pi(j)|)$ time for relevant colors i and j and decreases Φ_π by at least one, where π is the initial edge coloring and $E_\pi(i)$ is the set of edges colored with i.

In this paper, we investigate the running time of a modified version of their algorithm in which the initial edge coloring is generated randomly according to the following rule: Randomly pick a color $i \in \mathcal{C}$ to assign an edge $e \in E$ until all the edges are colored. Using Chernoff bound [4], we show that, for arbitrary constants $\gamma \in (0, 1)$ and $\varepsilon \in (0, 1/2)$, with high probability for sufficiently large n, such a random color assignment π satisfies $|E_\pi(i)| \le 2m/k$ for all colors $i \in \mathcal{C}$ and $\Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$ if $k = O(m^{1-\gamma})$. Hence, by repeatedly using Recolor, the random color assignment can be modified to a nearly equitable edge coloring in $O\left(n^{1/2}m^{3/2+\varepsilon}\right)$ time with high probability for sufficiently large n. This time complexity is better than Xie at al.'s original algorithm when the graph is dense and k is small.

The rest of the paper is organized as follows. In Section 2, we give some definitions. Section 3 introduces the new version of Xie et al.'s algorithm when it starts with a random color assignment, and Section 4 analyzes its time complexity. Finally, concluding remarks are in Section 5.

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2 Preliminaries

A multigraph G is a graph that allows multiple number of edges between vertices. Let V and E denote the sets of vertices and edges of G, respectively. The following definitions will be used throughout this paper.

- Let $n = |V|$ and $m = |E|$. Note that $m \ge n - 1$ holds for any connected graph.
- We denote the given color set by $\mathcal{C} = \{1, 2, \ldots, k\}$, where k is the number of given colors.
- We denote an edge coloring by a mapping π: $E \to \mathcal{C}$; i.e., if an edge $e \in E$ is colored with a color i, then $\pi(e) = i$.
- For each vertex $v \in V$, let $N(v) = \{(v, w) \in E\}$ denote the edges incident to v in G and $d(v) = |N(v)|$ be its degree. Then, $d_\pi(v, i) = |\{e \in N(v) \mid \pi(e) = i\}|$ stands for the number of edges colored with i and incident to v, while $E_\pi(i) = \{e \in E \mid \pi(e) = i\}$ stands for the set of edges in E colored with i.
- Let $V_\pi(i, j)$ be the set of end vertices of edges in $E_\pi(i) \cup E_\pi(j)$; i.e., $V_\pi(i, j) = \{v \in V \mid \exists w \in V,\ (v, w) \in (E_\pi(i) \cup E_\pi(j))\}$. Then let $G_\pi(i, j) = (V_\pi(i, j), E_\pi(i) \cup E_\pi(j))$ be the subgraph whose edges are $E_\pi(i) \cup E_\pi(j)$ and vertices are their end vertices.

Then, the definition of nearly equitable edge coloring introduced by Hilton and Werra [1] is as follows.

Definition 1 [1]: *Given a multigraph $G = (V, E)$ and a color set $\mathcal{C} = \{1, 2, \ldots, k\}$, the nearly equitable edge coloring π is an assignment of the given k colors to all the edges in G, such that for any vertex $v \in V$ and different colors $i, j \in \mathcal{C}$, $|d_\pi(v, i) - d_\pi(v, j)| \le 2$.*

Without loss of generality, we assume that G is connected.

3 The Algorithm Starting with a Random Color Assignment

We describe the modified version of Xie et al.'s algorithm, which starts with a random color assignment and calls algorithm Recolor [3] until a nearly equitable edge coloring is obtained. We call it algorithm RCAR, which stands for *Random Color Assignment and Recolor*.

The recoloring phase works as follows. Whenever the current coloring π has a vertex u that breaks the condition of Definition 1, choose two colors α and β with maximum and minimum $d_\pi(u, i)$, respectively, and call algorithm Recolor to recolor those edges in $E_\pi(\alpha) \cup E_\pi(\beta)$. Algorithm Recolor first constructs an augmented graph $\hat{G} = (\hat{V}, \hat{E})$ by adding a vertex and some edges to make $G_\pi(\alpha, \beta)$ connected and the degrees of all vertices even. It then finds an Euler circuit in $\hat{G}$ starting at the additional vertex and colors the edges alternately with α

and β along the Euler circuit, so that the resulting coloring π' is balanced with respect to the two colors, i.e., $||E_{\pi'}(\alpha)| - |E_{\pi'}(\beta)|| \leq 1$ holds after recoloring. The algorithm is formally described as follows.

Algorithm RCAR$(G, \mathcal{C})$
Input: a multigraph $G = (V, E)$ and a k-color set $\mathcal{C} = \{1, 2, \ldots, k\}$
Output: a nearly equitable edge coloring π for G

1. For each $e \in E$, randomly pick a color $i \in \mathcal{C}$ (the probability of choosing a color i is $1/k$) and let $\pi(e) \leftarrow i$.
2. **while** there exists $u \in V$ and different $i, j \in \mathcal{C}$ such that $|d_\pi(u,i) - d_\pi(u,j)| \geq 3$ **do**
3. For the vertex u, find two colors $\alpha, \beta \in \mathcal{C}$ satisfying $d_\pi(u,\alpha) = \max_{i\in\mathcal{C}} (d_\pi(u,i))$, $d_\pi(u,\beta) = \min_{i\in\mathcal{C}} (d_\pi(u,i))$.
4. Call RECOLOR(G, α, β, π) to modify π.
5. Output π and stop.

Algorithm RECOLOR(G, α, β, π)
Input: a multigraph $G = (V, E)$, colors α and β, and a coloring π
Task: modify the edge coloring π for $G_\pi(\alpha, \beta)$

1. Let $\hat{V} \leftarrow V_\pi(\alpha,\beta) \cup \{\hat{v}\}$ $(\hat{v} \notin V)$ and $\hat{E} \leftarrow E_\pi(\alpha) \cup E_\pi(\beta)$.
2. **for** each connected component H in $G_\pi(\alpha,\beta)$ **do**
3. **if** H has odd-degree vertices **then**
4. For each odd-degree vertex v in H, add an edge $(v, \hat{v})$ into $\hat{E}$.
5. **else**
6. **if** H has a vertex v such that $|d_\pi(v,\alpha) - d_\pi(v,\beta)| \geq 2$ **then** Draw two parallel edges between v and $\hat{v}$, and add them into $\hat{E}$.
7. **else** Let v be an arbitrary vertex in H. Draw two parallel edges between v and $\hat{v}$, and add them into $\hat{E}$.
8. Let $\hat{G} \leftarrow (\hat{V}, \hat{E})$.
9. Let a sequence of edges $e_1, e_2, \ldots, e_l$ be an Euler circuit of $\hat{G}$ such that the tail of e_1 is $\hat{v}$. Then let $\hat{\pi}(e_t) \leftarrow \alpha$ if t is odd and $\hat{\pi}(e_t) \leftarrow \beta$ otherwise for all $t = 1, 2, \ldots, l$.
10. Let $\pi(e) \leftarrow \hat{\pi}(e)$ for all edge e in $G_\pi(\alpha,\beta)$, and stop.

4 Analysis of Algorithm RCAR

4.1 Results of RECOLOR

Recoloring edges along with traversals of Euler circuits in graphs is a common technique for edge coloring. Xie et al. [3] shows that the running time of modifying an arbitrary initial edge coloring π to a nearly equitable edge coloring is decided by the running time of algorithm RECOLOR and the number of calls to it. To analyze the number of calls to RECOLOR, they introduce the following potential Φ_π and proved the following lemmas. For all vertices $v \in V$, let $\bar{d}(v) = \lfloor d(v)/k \rfloor$. Define

$$\Phi_\pi(v) = \sum_{i\in\mathcal{C}} \varphi^2_{\bar{d}(v)-1}(d_\pi(v,i))$$

$$\Phi_\pi = \sum_{v\in V} \Phi_\pi(v),$$

where $\varphi^2_{\bar{d}(v)-1}(d_\pi(v,i))$ is defined by

$$\varphi^b_a(x) = \max\{x - a - b, a - x, 0\} \tag{1}$$

with $x = d_\pi(v,i)$, $a = \bar{d}(v) - 1$ and $b = 2$. By definition, $\Phi_\pi \geq 0$ holds for any coloring π.

Lemma 1 [3]: *Let π and π' be the coloring before and after calling* RECOLOR. *Then* RECOLOR *runs in $O(|E_\pi(\alpha) \cup E_\pi(\beta)|)$ time and the coloring π' satisfies $||E_{\pi'}(\alpha)| - |E_{\pi'}(\beta)|| \leq 1$.*

Lemma 2 [3]: *Assume that there exists a vertex u and colors α, β such that $d_\pi(u,\alpha) \geq \bar{d}(u)+1$, $d_\pi(u,\beta) \leq \bar{d}(u)$ and $d_\pi(u,\alpha) - d_\pi(u,\beta) \geq 3$ for a coloring π, and let π' be the coloring after calling* RECOLOR(G,α,β,π). *Then $\Phi_{\pi'} \leq \Phi_\pi - 1$ holds.*

4.2 Analysis of Random Color Assignment

In this section, we use Chernoff bound [4] to show that, for arbitrary constants $\gamma \in (0,1)$ and $\varepsilon \in (0, 1/2)$, the random color assignment π of algorithm RCAR satisfies $|E_\pi(i)| \leq 2m/k$ for all colors $i \in \mathcal{C}$ and $\Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$ with high probability for sufficiently large n if $k = O(m^{1-\gamma})$. The explanation of Chernoff bound can be found in many books such as [5, 6]. Here we use the following two versions of it in Section 4.1 of [5] and in Appendix A of [6], respectively.

Theorem 1 [4, 5]: *Let $X_1, X_2, \ldots, X_r$ be independent Poisson trials such that, for $1 \leq i \leq r$, $\Pr(X_i = 1) = p_i$ and $\Pr(X_i = 0) = 1 - p_i$, where $0 < p_i < 1$. Then, for $X = \sum_{i=1}^r X_i$, $\mu = \sum_{i=1}^r p_i$, and any $\delta > 0$,*

$$\Pr(X > (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu. \tag{2}$$

Theorem 2 [4, 6]: *Let $Y_1, Y_2, \ldots, Y_r$ be mutually independent random variables such that, for $1 \leq i \leq r$, $\Pr(Y_i = 1 - p_i) = p_i$ and $\Pr(Y_i = -p_i) = 1 - p_i$, where $0 \leq p_i \leq 1$. Then, for $Y = \sum_{i=1}^r Y_i$ and any $a > 0$, $\Pr(Y > a) < e^{-2a^2/r}$ holds. Note that $\Pr(Y < -a) < e^{-2a^2/r}$ also holds by symmetry, and hence we have*

$$\Pr(|Y| > a) < 2e^{-2a^2/r}. \tag{3}$$

Each edge $e \in E$ is colored with a color $i \in \mathcal{C}$ with probability $1/k$. For each $e \in E$ and $i \in \mathcal{C}$, we define an indicator random variable $X^e_\pi(i)$ that takes value 1 if and only if e is colored with i. Then we have

$$X^e_\pi(i) = \begin{cases} 1 & \text{with probability } \frac{1}{k} \\ 0 & \text{with probability } 1 - \frac{1}{k}. \end{cases}$$

For the random coloring, $|E_\pi(i)|$ is a random variable satisfying

$$|E_\pi(i)| = \sum_{e\in E} X^e_\pi(i). \tag{4}$$

Using Chernoff bound (2) with $\delta = 1$ and $\mu = m/k$, for all colors $i \in \mathcal{C}$, we have

$$\Pr\left(|E_\pi(i)| > \frac{2m}{k}\right) < \left(\frac{e}{4}\right)^{m/k}. \qquad (5)$$

For $k \le cm^{1-\gamma}$ ($c > 0$ and $\gamma \in (0,1)$ are constants), the probability of the existence of a color $i \in \mathcal{C}$ satisfying $|E_\pi(i)| > 2m/k$ is

$$\Pr\left(\exists i, |E_\pi(i)| > \frac{2m}{k}\right) \le \sum_{i\in\mathcal{C}} \Pr\left(|E_\pi(i)| > \frac{2m}{k}\right) \le k\left(\frac{e}{4}\right)^{m/k} \le cm^{1-\gamma}\left(\frac{e}{4}\right)^{m^\gamma/c}. \qquad (6)$$

Thus, we obtain the following lemma.

Lemma 3: *Given an n-vertex m-edge multigraph $G = (V, E)$ and a k-color set $\mathcal{C} = \{1, 2, \ldots, k\}$, for any constants $c > 0$ and $\gamma \in (0, 1)$, the random coloring π satisfies $|E_\pi(i)| \le 2m/k$ for all colors $i \in \mathcal{C}$ with probability at least $1 - cm^{1-\gamma}(e/4)^{m^\gamma/c}$ if $k \le cm^{1-\gamma}$.*

We now consider the bound on Φ_π of the random coloring. To make the proofs in the following simple, we define $\hat{\Phi}_\pi(v)$ and $\hat{\Phi}_\pi$ as follows:

$$\begin{aligned}\hat{\Phi}_\pi(v) &= \sum_{i\in\mathcal{C}} \varphi^0_{d(v)/k}\left(d_\pi(v,i)\right) \\ &= \sum_{i\in\mathcal{C}} \left|d_\pi(v,i) - \frac{d(v)}{k}\right| \\ \hat{\Phi}_\pi &= \sum_{v\in V} \hat{\Phi}_\pi(v),\end{aligned}$$

where $\varphi^0_{d(v)/k}\left(d_\pi(v,i)\right)$ is defined by equation (1) with $x = d_\pi(v,i)$, $a = d(v)/k$ and $b = 0$. It is easy to show that for any constants a, a', $b \ge 0$ and $b' \ge 0$ satisfying $a \le a'$ and $a' + b' \le a + b$, $\varphi^b_a(x) \le \varphi^{b'}_{a'}(x)$ holds for all x. Thus, for $\bar{d}(v) - 1 \le d(v)/k$ and $(d(v)/k) + 0 \le (\bar{d}(v) - 1) + 2$, we obtain $\Phi_\pi(v) \le \hat{\Phi}_\pi(v)$ for all vertices $v \in V$ and $\Phi_\pi \le \hat{\Phi}_\pi$.

Let $\varepsilon \in (0, 1/2)$ be an arbitrary constant. For the vertices $v \in V$ with $d(v) \le n^\varepsilon$, we obtain

$$\begin{aligned}\hat{\Phi}_\pi(v) &= \sum_{\substack{i\in\mathcal{C} \\ d_\pi(v,i) > d(v)/k}} \left(d_\pi(v,i) - \frac{d(v)}{k}\right) + \sum_{\substack{i\in\mathcal{C} \\ d_\pi(v,i) \le d(v)/k}} \left(\frac{d(v)}{k} - d_\pi(v,i)\right) \\ &\le \sum_{\substack{i\in\mathcal{C} \\ d_\pi(v,i) > d(v)/k}} d_\pi(v,i) + \sum_{\substack{i\in\mathcal{C} \\ d_\pi(v,i) \le d(v)/k}} \frac{d(v)}{k} \\ &\le 2d(v) = O(n^\varepsilon).\end{aligned}$$

Hence, the contribution of such vertices to the value of $\hat{\Phi}_\pi$ is

$$\sum_{v:\, d(v) \le n^\varepsilon} \hat{\Phi}_\pi(v) = O(n^{\varepsilon+1}). \qquad (7)$$

For the remaining vertices $v \in V$ with $d(v) > n^\varepsilon$, we use Chernoff bound (3) to show their contribution to the value of $\hat{\Phi}_\pi$. For convenience, we define a random variable $Y^e_\pi(i) = X^e_\pi(i) - 1/k$, which satisfies

$$Y^e_\pi(i) = \begin{cases} 1 - \frac{1}{k} & \text{with probability } \frac{1}{k} \\ -\frac{1}{k} & \text{with probability } 1 - \frac{1}{k}. \end{cases}$$

Then, let

$$Y_\pi(v,i) = \sum_{e\in N(v)} Y^e_\pi(i),$$

which signifies the number of the edges incident to a vertex v and colored with i. Then, $d_\pi(v,i)$ becomes a random variable, which satisfies

$$\begin{aligned}d_\pi(v,i) &= \sum_{e\in N(v)} X^e_\pi(i) = \sum_{e\in N(v)} \left(Y^e_\pi(i) + \frac{1}{k}\right) \\ &= \sum_{e\in N(v)} Y^e_\pi(i) + \frac{d(v)}{k} = Y_\pi(v,i) + \frac{d(v)}{k}.\end{aligned}$$

Using Chernoff bound (3), we obtain

$$\begin{aligned}&\Pr\left(\left|d_\pi(v,i) - \frac{d(v)}{k}\right| > (d(v))^{1/2+\varepsilon}\right) \\ &= \Pr(|Y_\pi(v,i)| > (d(v))^{1/2+\varepsilon}) \\ &< 2e^{-2(d(v))^{2\varepsilon}} < 2e^{-2n^{2\varepsilon^2}} \qquad (8)\end{aligned}$$

for all vertices $v \in V$ with $d(v) > n^\varepsilon$ and all colors $i \in \mathcal{C}$. Thus, the probability that there exists a pair of $v \in V$ with $d(v) > n^\varepsilon$ and $i \in \mathcal{C}$ satisfying $|d_\pi(v,i) - d(v)/k| > (d(v))^{1/2+\varepsilon}$ is

$$\begin{aligned}&\Pr\left(\exists v \text{ s.t. } d(v) > n^\varepsilon, \exists i, \left|d_\pi(v,i) - \frac{d(v)}{k}\right| > (d(v))^{1/2+\varepsilon}\right) \\ &< \sum_{i\in\mathcal{C}} \sum_{v:\, d(v) > n^\varepsilon} 2e^{-2n^{2\varepsilon^2}} \le 2kne^{-2n^{2\varepsilon^2}}. \qquad (9)\end{aligned}$$

Then, we have

$$\forall v \text{ s.t. } d(v) > n^\varepsilon, \forall i, \left|d_\pi(v,i) - \frac{d(v)}{k}\right| \le (d(v))^{1/2+\varepsilon}$$

with probability at least $1 - 2kne^{-2n^{2\varepsilon^2}}$. When this happens, we obtain

$$\begin{aligned}\sum_{v:\, d(v) > n^\varepsilon} \hat{\Phi}_\pi(v) &\le k \sum_{v:\, d(v) > n^\varepsilon} (d(v))^{1/2+\varepsilon} \\ &\le k \sum_v (d(v))^{1/2+\varepsilon} \\ &\le kn\left(\frac{2m}{n}\right)^{1/2+\varepsilon} \\ &= O(kn^{1/2-\varepsilon} m^{1/2+\varepsilon}). \qquad (10)\end{aligned}$$

Note that the third inequality $\sum_v (d(v))^{1/2+\varepsilon} \le n\,(2m/n)^{1/2+\varepsilon}$ follows from the fact that $(d(v))^{1/2+\varepsilon}$ is a concave function of $d(v)$ for any $\varepsilon \in (0, 1/2)$ and $\sum_{v\in V} d(v) = 2m$ holds.

Then, (7) and (10) imply

$$\begin{aligned} \Phi_\pi \le \hat{\Phi}_\pi &= O(n^{\varepsilon+1} + kn^{1/2-\varepsilon}m^{1/2+\varepsilon}) \\ &= O(kn^{1/2}m^{1/2+\varepsilon}) \end{aligned} \tag{11}$$

(recall that we assumed G is connected, which implies $m \ge n-1$). Thus, we obtain the following lemma.

Lemma 4: *Given an n-vertex m-edge multigraph $G = (V, E)$ and a k-color set $\mathcal{C} = \{1, 2, \ldots, k\}$, for any constant $\varepsilon \in (0, 1/2)$, the random coloring π satisfies $\Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$ with probability at least $1 - 2kne^{-2n^{2\varepsilon^2}}$.*

Using Lemmas 3 and 4, we have

$$\forall i, |E_\pi(i)| \le \frac{2m}{k} \text{ and } \Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$$

with probability at least $1 - cm^{1-\gamma}(e/4)^{m^\gamma/c} - 2kne^{-2n^{2\varepsilon^2}}$. For

$$\lim_{n\to\infty}\left(1 - cm^{1-\gamma}\left(\frac{e}{4}\right)^{m^\gamma/c} - 2kne^{-2n^{2\varepsilon^2}}\right) = 1,$$

we have the following lemma about the random coloring.

Lemma 5: *Given an n-vertex m-edge multigraph $G = (V, E)$ and a k-color set $\mathcal{C} = \{1, 2, \ldots, k\}$, randomly pick a color $i \in \mathcal{C}$ to assign an edge $e \in E$ until all the edges are colored. Then, for any constants $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, 1)$, such a random coloring π satisfies $|E_\pi(i)| \le 2m/k$ for all colors $i \in \mathcal{C}$ and $\Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$ almost surely for sufficiently large n if $k = O(m^{1-\gamma})$.*

4.3 Running Time of Algorithm RCAR

We now consider the total running time of algorithm RCAR. For arbitrary constants $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, 1)$, assume that $\max_{i\in\mathcal{C}} |E_\pi(i)| \le 2m/k$ and $\Phi_\pi = O(kn^{1/2}m^{1/2+\varepsilon})$ hold for the initial random coloring, which are satisfied almost surely for sufficiently large n if $k = O(m^{1-\gamma})$ by Lemma 5. Then, Lemma 1 implies that RECOLOR runs in $O(m/k)$ time and $\max_{i\in\mathcal{C}} |E_{\pi'}(i)| \le 2m/k$ holds for the coloring π' after executing RECOLOR. Whenever RECOLOR is called from algorithm RCAR, the vertex u and colors α and β in Line 3 of RCAR satisfy $d_\pi(u, \alpha) \ge \bar{d}(u) + 1$, $d_\pi(u, \beta) \le \bar{d}(u)$ and $d_\pi(u, \alpha) - d_\pi(u, \beta) \ge 3$. Then, using Lemma 2, Φ_π decreases by at least 1 when invoking RECOLOR. Recall that $\Phi_\pi \ge 0$ holds for any coloring π. Hence the number of calls to RECOLOR cannot exceed the initial value of Φ_π. The above assumption implies that, after $O(kn^{1/2}m^{1/2+\varepsilon})$ invocations of RECOLOR, algorithm RCAR must stop. Thus, the total running time of algorithm RCAR becomes

$$O\left(kn^{1/2}m^{1/2+\varepsilon}\left(\frac{m}{k}\right)\right) = O\left(n^{1/2}m^{3/2+\varepsilon}\right), \tag{12}$$

and we obtain the following theorem.

Theorem 3: *For any constants $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, 1)$, algorithm RCAR solves the nearly equitable edge coloring problem in $O\left(n^{1/2}m^{3/2+\varepsilon}\right)$ time almost surely for sufficiently large n if the number of given colors $k = O(m^{1-\gamma})$, where m and n are the numbers of edges and vertices, respectively.*

Below we discuss when the above running time of algorithm RCAR becomes faster than $O(m^2/k)$, which is the fastest running time among the existing algorithms.

Consider the ratio

$$\frac{n^{1/2}m^{3/2+\varepsilon}}{m^2/k} = kn^{1/2}m^{-1/2+\varepsilon}. \tag{13}$$

The condition

$$\lim_{n\to\infty}\left(kn^{1/2}m^{-1/2+\varepsilon}\right) = 0 \tag{14}$$

is equivalent to

$$kn^{1/2} = o(m^{1/2-\varepsilon}). \tag{15}$$

This is satisfied when the graph is dense and k is small; e.g., when $k = O(1)$ and $m = \Omega(n^\theta)$ for a constant $\theta > 1$.

5 Concluding remarks

In this paper, we considered the nearly equitable edge coloring problem and introduced an algorithm called RCAR, which is a modified version of Xie et al.'s algorithm [3] in that it starts with a random color assignment. For any constants $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, 1)$, algorithm RCAR runs in $O\left(n^{1/2}m^{3/2+\varepsilon}\right)$ time with high probability for large n if the number of given colors $k = O(m^{1-\gamma})$, where n and m are the numbers of vertices and edges, respectively. This computation time is faster than the existing algorithms when the graph is dense and k is small.

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